MMAT5390: Mathematical Image Processing

Assignment 1 Solutions

1. Write H as a block matrix $\begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}$.

$$\begin{aligned} C_{ij} &= \begin{pmatrix} h^{1,i}(1,j) & h^{1,i}(2,j) & \cdots & h^{1,i}(n,j) \\ h^{2,i}(1,j) & h^{2,i}(2,j) & \cdots & h^{2,i}(n,j) \\ \vdots & \vdots & \ddots & \vdots \\ h^{n,i}(1,j) & h^{n,i}(2,j) & \cdots & h^{n,i}(n,j) \end{pmatrix} \\ &= \begin{pmatrix} A_{11}B_{ji} & A_{12}B_{ji} & \cdots & A_{1n}B_{ji} \\ A_{21}B_{ji} & A_{22}B_{ji} & \cdots & A_{2n}B_{ji} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B_{ji} & A_{n2}B_{ji} & \cdots & A_{nn}B_{ji} \end{pmatrix} \\ &= B_{ji}A \end{aligned}$$

Hence,

$$H = \begin{pmatrix} B_{11}A & B_{21}A & \cdots & B_{n1}A \\ B_{12}A & B_{22}A & \cdots & B_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ B_{n2}A & B_{n2}A & \cdots & B_{nn}A \end{pmatrix} = B^T \otimes A.$$

2. Write H as a block matrix $\begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,n} \end{pmatrix}$.

Note that H is block-circulant if $C_{i,j} = C_{i+1,j+1}$. Since H is the transformation matrix of a shift-invariant linear image transformation, we have $h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$.

$$\begin{aligned} C_{i+1,j+1} &= \begin{pmatrix} h^{1,i+1}(1,j+1) & h^{1,i+1}(2,j+1) & \cdots & h^{1,i+1}(n,j+1) \\ h^{2,i+1}(1,j+1) & h^{2,i+1}(2,j+1) & \cdots & h^{2,i+1}(n,j+1) \\ \vdots & \vdots & \ddots & \vdots \\ h^{n,i+1}(1,j+1) & h^{n,i+1}(2,j+1) & \cdots & h^{n,i+1}(n,j+1) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{h}(1-1,i+1-(j+1)) & \tilde{h}(1-2,i+1-(j+1)) & \cdots & \tilde{h}(1-n,i+1-(j+1)) \\ \tilde{h}(2-1,i+1-(j+1)) & \tilde{h}(2-2,i+1-(j+1)) & \cdots & \tilde{h}(2-n,i+1-(j+1)) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}(n-1,i+1-(j+1)) & \tilde{h}(n-2,i+1-(j+1)) & \cdots & \tilde{h}(n-n,i+1-(j+1)) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{h}(1-1,i-j) & \tilde{h}(1-2,i-j) & \cdots & \tilde{h}(1-n,i-j) \\ \tilde{h}(2-1,i-j) & \tilde{h}(2-2,i-j) & \cdots & \tilde{h}(2-n,i-j) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}(n-1,i-j) & \tilde{h}(n-2,i-j) & \cdots & \tilde{h}(n-n,i-j) \end{pmatrix} \\ &= \begin{pmatrix} h^{1,i}(1,j) & h^{1,i}(2,j) & \cdots & h^{1,i}(n,j) \\ h^{2,i}(1,j) & h^{2,i}(2,j) & \cdots & h^{2,i}(n,j) \\ \vdots & \vdots & \ddots & \vdots \\ h^{n,i}(1,j) & h^{n,i}(2,j) & \cdots & h^{n,i}(n,j) \end{pmatrix} \\ &= C_{i,j} \end{aligned}$$

Hence, H is block-circulant.

3. (a)

$$
\begin{aligned}
H &= B^T \otimes A \\
&= \begin{pmatrix} A & 0 & 2A \\ 2A & 0 & A \\ 0 & 3A & A \end{pmatrix} \\
&= \begin{pmatrix}
1 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 6 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\
7 & 0 & 9 & 0 & 0 & 0 & 14 & 0 & 18 \\
2 & 0 & 6 & 0 & 0 & 0 & 1 & 0 & 3 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
14 & 0 & 18 & 0 & 0 & 0 & 7 & 0 & 9 \\
0 & 0 & 0 & 3 & 0 & 9 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 15 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 21 & 0 & 27 & 7 & 0 & 9
\end{pmatrix}
\end{aligned}
$$

(b) i. If $\mathcal{O}$ is separable, then $H = B^T \otimes A$ for some $A, B \in M_{22}$.
Write $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$. Then H_{ij} is a scalar multiple of A for all i, j.
However, $H_{12}(1,1) = \pi^3 H_{11}(1,1)$ and $H_{12}(1,2) = \pi H_{11}(1,2) \neq \pi^3 H_{11}(1,2)$.
Therefore, $\mathcal{O}$ is not separable.
Notice that $\mathcal{O}$ is shift-invariant if and only if H is block-circulant and each block matrix is circulant. Although $H = \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}$ is block-circulant, $H_1 = \begin{pmatrix} \pi & \pi^2 \\ \pi^3 & \pi^4 \end{pmatrix}$ is not circulant.
Hence, $\mathcal{O}$ is not shift-invariant.

ii. Note that $H = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 0 & 2 \\ 1 & 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = B^T \otimes A$,
where $A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 4 \end{pmatrix}$.
Therefore, $\mathcal{O}(f) = AfB$ is separable. Also note that H is not block-circulant since $H_{11} \neq H_{22}$. $\mathcal{O}$ is not shift-invariant.
(Remark: There are two typos in matrix H (row 6, col 8 and row 9, col 5). So, marks will be given for both of the following: 1. correctly explaining if your answer is separable; 2. your answer is not separable.)

4. Assume f and k are periodically extended.
Suppose $\mathcal{O}(f) = k * f$ for some $k \in M_{N \times N}(\mathbb{R})$. Then,

$$
\begin{aligned}
\mathcal{O}(f)(\alpha, \beta) &= k * f(\alpha, \beta) \\
&= \sum_{x=1}^{N} \sum_{y=1}^{N} k(x, y) f(\alpha - x, \beta - y) \\
&= \ldots + k(-1,0)f(\alpha+1,\beta) + k(0,0)f(\alpha,\beta) + k(1,0)f(\alpha-1,\beta) + k(0,-1)f(\alpha,\beta+1) \\
&\quad + k(0,1)f(\alpha,\beta-1) + \ldots
\end{aligned}
$$

Since we know that

$$
O(f)(\alpha, \beta) = \frac{1}{4}\left[-5f(\alpha,\beta) + 2f(\alpha+1,\beta) + 3f(\alpha-1,\beta) + 4f(\alpha,\beta+1) + f(\alpha,\beta-1)\right].
$$

By setting $k(-1,0) = k(N-1,N) = \frac{1}{2}$, $k(0,0) = k(N,N) = -\frac{5}{4}$, $k(1,0) = k(1,N) = \frac{3}{4}$, $k(0,-1) = k(N,N-1) = 1$, $k(0,1) = k(N,1) = \frac{1}{4}$ and set other entries of k equal to 0 otherwise. $\mathcal{O}(f)$ is then equal to $k * f$.

5. (a)

$$
f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}
$$

$$f^T f = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{pmatrix}$$

The eigenvalues are:

$$\lambda_1 = 36, \quad \lambda_2 = 3, \quad \lambda_3 = 3$$

Thus, the singular values are:

$$\sigma_1 = \sqrt{\lambda_1} = 6, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{3}, \quad \sigma_3 = \sqrt{\lambda_3} = \sqrt{3}$$

$$\Sigma = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

For each eigenvalue of $f^T f$, the corresponding unit eigenvectors are:

$$v_1 = \frac{1}{\sqrt{3}}(1,1,1)^T, \quad v_2 = \frac{1}{\sqrt{2}}(1,-1,0)^T, \quad v_3 = \frac{1}{\sqrt{6}}(1,1,-2)^T$$

(Remark: The above eigenvectors are not the only option.)
Construct the V matrix:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

Using $u_i = \frac{f v_i}{\sigma_i}$, we get:

$$u_1 = \frac{f v_1}{6} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$$

$$u_2 = \frac{f v_2}{\sqrt{3}} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$$

$$u_3 = \frac{f v_3}{\sqrt{3}} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$$

Construct the U matrix:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Finally, the SVD of f is:

$$f = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}^T$$

(b)

$$f = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T = 6 \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + \sqrt{3} \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

(c) Note that $f^T f = g^T g$, so the matrix Σ for the SVD of f and g are the same. Now we can use the same matrix V to compute the SVD of g, but the matrix U we obtained for g is different:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$